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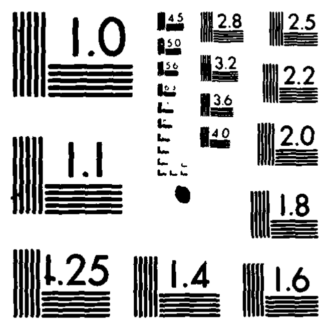


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# UNIFORM ASYMPTOTIC EXPANSIONS FOR UNSTEADY FLOW IN UNCONFINED AQUIFERS


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# UNIFORM ASYMPTOTIC EXPANSIONS FOR UNSTEADY FLOW IN UNCONFINED AQUIFERS

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ABSTRACT

A general uniform asymptotic expansion is constructed to remove certain anomalies which appear in the ray method expansion for unsteady flow in unconfined aquifers, and makes the asymptotic method based upon the principles of geometrical optics even more useful. In particular, expansions uniform in a region containing a caustic, a line of zero depth, a combination of both, two caustics, and two lines of zero depth are explicitly constructed.

AMS (MOS) Subject Classification: 76S05

Key Words: Unconfined aquifers, Uniform asymptotic expansions, Caustics, lines of zero depth

Work Unit Number 2 - Physical Mathematics



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# UNIFORM ASYMPTOTIC EXPANSIONS FOR UNSTEADY FLOW IN UNCONFINED AQUIFERS

M. C. Shen

## I. Introduction

The problem of fluid flow in unconfined aquifers has attracted much attention in recent years, because of its great importance in underground water flow, land subsidence, oil flow to wells, and many others. Several approximate theories for the study of such a problem are now at our disposal, for example, the linear equations, the Boussinesq equation (Scheidegger [1]), and the delayed field equation (Boulton [2], Herrera et al [3]). However, these equations are generally used to deal with homogeneous aquifers of uniform depth only. In a previous paper (Shen [4]), we developed an asymptotic method employing rays as in geometrical optics for the study of underground flow in unconfined nonhomogeneous aquifers of variable depth governed by linearized equations. It was motivated by the work on diffusion equation (Cohen and Lewis [5], Voronka and Keller [6]), and on surface waves on a fluid of variable depth (Keller [7], Shen [8]). The asymptotic expansion for the solution of the problem consists of a phase function and successive approximations to an amplitude function. The phase function satisfies the Hamilton-Jacobi equation and may be solved by the method of characteristics, which in turn determines a family of curves called rays. The successive approximations to the amplitude function satisfy a transport equation along the rays and can be determined by simple integration. However, the ray method fails at the places where the rays form an envelope or the depth of the aquifer is zero, and the amplitude function becomes infinite there.

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The purpose of this work is to remove these anomalies by means of uniform asymptotic expansions in terms of solutions of general second order ordinary differential equations in a manner as suggested in Shen and Keller [9]. By appropriately choosing the coefficients in the differential equation, and using the solution of the equation in the asymptotic expansion, the amplitude function then becomes finite at these anomalies. The refined method is no more difficult to use than the ray method, since by a simple transformation the results in the former are reduced to those in the latter. The uniform asymptotic expansion at a caustic is a slight modification of the method due to Ludwig [10]. However, the behavior of a solution at a line of zero depth of an unconfined aquifer is studied here for the first time.

We formulate the problem and recapitulate the ray method in §2. In §3 we consider the uniform asymptotic expansion under the shallow-water assumption, which is simpler to use. In §4, we consider several special cases, for which comparison equations are explicitly constructed. Some of the detailed derivations are given in the appendices.

## 2. Formulation

The linearized equations governing the compressible flow in an unconfined aquifer are assumed to be the following:

$$\nabla^* \cdot (K^* \nabla^* \phi^*) = S_1^* \phi^*_{t^*} \quad , \tag{1}$$

$$S_2^* \phi^*_{t^*} + K^* \phi^*_{z^*} = 0 \quad \text{at} \quad z^* = 0 \quad , \tag{2}$$

$$\nabla^* \phi^* \cdot \nabla^*(z^* + h^*) = 0 \quad \text{at} \quad z^* = -h^* \quad . \tag{3}$$

The physical meaning and derivation of (1) to (3) may be found in Scheidegger [1]. Here $\phi^*$ is the potential, $\nabla^* = (\partial/\partial x^*, \partial/\partial y^*, \partial/\partial z^*)$, $z^*$ is positive upward, $K^*(x^*,y^*,z^*)$, $S_1^*(x^*,y^*,z^*)$, and $S_2^*(x^*,y^*)$, are given positive functions assumed to be sufficiently smooth. The free surface of the flow is $z^* = \eta^*(x^*,y^*,t^*) = -\phi^*/g$ at $z^* = 0$ where $g^*$ is the constant gravitational acceleration, and $z^* = -h^*$ is the rigid bottom, nonpositive and sufficiently smooth. We measure $x^*$, $y^*$, $z^*$ and $h^*$ in units of $H$, the mean depth of the aquifer, $t^*$ in units of $(H/g)^{1/2}$. Assume that $K^*(x^*,y^*,z^*)$ is bounded below by a positive constant $M$ and measure $K^*$ in units of $M$. Now let the horizontal length scale be $L$ and assume $\beta^{3/2} = L/H$ where $\beta$ is a large parameter. The nondimensional variables are

$$x = \beta^{-3/2} x^*/H, \quad y = \beta^{-3/2} y^*/H, \quad h = h^*/H \quad ,$$

$$z = z^*/H, \quad K = K^*/M, \quad t = \beta^{-2} t^*/(H/g)^{1/2} \quad ,$$

$$S_1 = S_1^* H^{3/2} g^{1/2}/M, \quad S_2 = S_2^*(g/H)^{1/2}/M, \quad \phi = \phi^*/(gH) \quad .$$

In terms of them, (1) to (3) become

$$\nabla \cdot (K \nabla \phi) + \beta^3 (K\phi_z)_z = S_1 \beta \phi_t \quad , \tag{4}$$

$$S_2 \phi_t + \beta^2 K \phi_z = 0 \quad \text{at} \quad z = 0 \quad , \tag{5}$$

$$\nabla \phi \cdot \nabla h + \beta^2 \phi_z = 0 \quad \text{at} \quad z = h \quad , \tag{6}$$

where $\nabla = (\partial/\partial x, \partial/\partial y)$.

Assume that (4) to (6) possess an asymptotic expansion of the form

$$\phi \sim \exp[-\beta\theta(x,y,t)](A_0 + \beta^{-1}A_1 + \beta^{-2}A_2 + \cdots\cdots) \quad . \tag{7}$$

Substitution of (7) in (4) to (6) yields a sequence of governing equations and boundary conditions for the successive approximations $A_0, A_1, A_2, \ldots$ and the results are summarized as follows. Their derivations may be found in Shen [4]. Let $\omega = -\theta_t$, $\bar{k} = \nabla\theta$, and $k^2 = (\nabla\theta)^2$. Then $\theta$ satisfies

$$\omega = Dk^2 \tag{8}$$

where

$$D = \int_{-h}^{0} K\,dz/I, \quad I = S_2 + \int_{-h}^{0} S_1\,dz \quad . \tag{9}$$

The characteristic equations for (8) (Courant and Hilbert [11]) are

$$\begin{aligned} &dt/d\sigma = \mu, \quad d\bar{r}/d\sigma = 2\mu\, D\bar{k}, \quad d\omega/d\sigma = 0 \quad , \\ &d\bar{k}/d\sigma = -\mu k^2 \nabla D, \quad d\theta/d\sigma = \mu\omega \quad , \end{aligned} \tag{10}$$

where $\bar{r} = (x,y)$ and $\mu$ is a proportional factor. The solutions of (10) determine a two-parameter family of time-space curves called rays, given by

$$\bar{r} = \bar{r}(\sigma,\sigma_1,\sigma_2), \; t = t(\sigma,\sigma_1,\sigma_2) \quad , \tag{11}$$

where $\sigma_1$, $\sigma_2$ are constant along each ray. By integrating $d\theta/d\sigma = \mu\omega$ along a ray, we have

$$\theta(\sigma) = \theta(\sigma_0) + \int_{\sigma_0}^{\sigma} \mu\omega d\sigma \quad ,$$

where $\theta(\sigma_0)$ is the value of $\theta$ at some initial point on the ray.

From the equations for the second approximation, we obtain

$$\begin{aligned} &-S_2\omega\, A_1(x,y,0,t) - S_2 A_{0t} - A_0 K(x,y,-h)\bar{k} \cdot \nabla h \\ &= \int_{-h}^{0} [(S_1\omega - Kk^2)A_1 + A_0\, \nabla \cdot K\bar{k} + 2K\, \bar{k} \cdot \nabla A_0 + S_1 A_{0t}]dz \quad , \end{aligned} \tag{12}$$

where

$$A_1 = A_0(-k^2 L + \omega M) + f(x,y,t) \quad ,$$

$$L(x,y,z) = \int_{-h}^{z} K^{-1}(x,y,z) \int_{-h}^{z} K(x,y,z')dz'dz \quad ,$$

$$M(x,y,z) = \int_{-h}^{z} K^{-1}(x,y,z) \int_{-h}^{z} S_1(x,y,z')dz'dz \quad .$$

By (8) and (10), (12) can be reduced to the following form

$$(IA_0^2)_t + \nabla \cdot (IA_0^2 \, d\bar{x}/dt) + 2NA_0^2 = 0 \quad , \tag{13}$$

where

$$N(x,y,t) = -k^2[\int_{-h}^{0} (S_1\omega - Kk^2)L(x,y,z)dz + S_2\omega L(x,y,0)]$$

$$+ \omega[\int_{-h}^{0} (S_1\omega - Kk^2)M(x,y,z)dz + S_2\omega M(x,y,0)] \quad .$$

(13) can be integrated along a ray to yield

$$IA_0^2 J(t)\exp(-\int_{t_0}^{t} 2NI^{-1}dt) = \text{constant} \quad , \tag{14}$$

where $J(t)$ is the Jacobian of transformation from the ray coordinates $\sigma = t, \sigma_1, \sigma_2$ to the coordinates $x, y, t$. At $h = 0$ and at a caustic where $J(t) = 0$, $A_0$ becomes infinite and the ray method fails. In the following section, we shall construct uniform asymptotic expansions to remove these anomalies.

## 3. Uniform Asymptotic Expansions

The main idea of constructing a uniform asymptotic expansion for (4) to (6) is based upon the assumptions that a reflected ray emerges after an incident ray reaches an anomaly, and the behavior of the solution there may be asymptotically approximated by the solution of an ordinary differential equation as a comparison equation. The ansatz is the following:

$$\phi = \exp(-\beta\theta)[\phi^{(1)}V(\beta^{\mu}\zeta) + \beta^{\mu-1}\phi^{(2)}V'(\beta^{\mu}\zeta)] \quad , \tag{15}$$

Here $V(\beta^{\mu}\zeta)$ satisfies

$$V''(\beta^{\mu}\zeta) + \beta^{-\mu}P(\zeta)V'(\beta^{\mu}\zeta) - \beta^{2-2\mu}Q^2(\zeta)V(\beta^{\mu}\zeta) = 0 \quad , \tag{16}$$

$$\phi^{(j)} \sim \phi_0^{(j)} + \beta^{-1}\phi_1^{(j)} + \beta^2\phi_2^{(j)} + \cdots \quad , \quad j = 1,2, \tag{17}$$

$\theta$ and $\zeta$ are functions of $x$, $y$ and $t$, $\mu$ is a real number, and $P(\zeta)$ and $Q(\zeta)$ are to be chosen.

We substitute (15) in (4) to (6), make use of (16), and equate the coefficients of $V$ and $V'$ to zero. Then by letting

$$\phi^{\pm} = \phi^{(1)} \pm Q\phi^{(2)} \quad , \tag{18}$$

$$S^{\pm} = \theta \mp \int_0^{\zeta} Q(\xi)d\xi \quad , \tag{19}$$

we obtain a sequence of equations and boundary conditions in terms of quantities defined above.

The equations for the zeroth approximations are

$$(K\phi_{0z}^{(1)})_z = (K\phi_{0z}^{(2)})_z = 0 \quad ,$$

$$\phi_{0z}^{(1)} = \phi_{0z}^{(2)} = 0 \quad \text{at} \quad z = 0, -h \quad .$$

It is easily seen that

$$(K\phi_{0z}^{\pm})_z = 0$$

$$\phi_{0z}^{\pm} = 0 \quad \text{at} \quad z = 0, -h \quad ,$$

and they imply that $\phi_0^{\pm}$ are functions of $x$, $y$ and $t$ only. The equations for the first approximations are

$$(K\phi_{1z}^{(1)})_z - K\,\nabla\theta \cdot [-\phi_0^{(1)}\nabla\theta + \phi_0^{(2)}Q^2\nabla\zeta] + KQ^2\nabla\zeta \cdot [-\nabla\theta\phi_0^{(2)} + \phi_0^{(1)}\nabla\zeta] = S_1[-\phi_0^{(1)}\theta_t + \phi_0^{(2)}\zeta_t Q^2] \quad , \tag{20}$$

$$(K\phi_{1z}^{(2)})_z + K[-\phi_0^{(1)}\nabla\zeta \cdot \nabla\theta + \phi_0^{(2)}Q^2(\nabla\zeta)^2] - K[-\phi_0^{(2)}(\nabla\theta)^2 + \phi_0^{(1)}\nabla\theta \cdot \nabla\zeta] = S_1[-\phi_0^{(2)}\theta_t + \phi_0^{(1)}\zeta t] \quad ; \tag{21}$$

at $z = 0$,

$$K\phi_{1z}^{(1)} + S_2[-\phi_0^{(1)}\theta_t + \phi^{(2)}\zeta_t Q^2] = 0 \quad , \tag{22}$$

$$K\phi_{1z}^{(2)} + S_2[-\phi_0^{(2)}\theta_t + \phi_0^{(1)}\zeta_t] = 0 \quad , \tag{23}$$

at $z = -h$,

$$\phi_{1z}^{(1)} = \phi_{1z}^{(2)} = 0 \quad . \tag{24}$$

By multiplying (21) by $\pm Q$ and adding the resulting equation to (20), we obtain

$$(K\phi_{1z}^{\pm})_z = [S_1\omega^{\pm} - K(k^{\pm})^2]\phi_0^{\pm} \quad , \tag{25}$$

where

$$\omega^{\pm} = -S_t^{\pm} = -(\theta_t \mp Q\zeta_t) \quad .$$

$$\bar{K}^{\pm} = \nabla S^{\pm} = \nabla\theta \mp Q\nabla\zeta \quad .$$

Similarly, (23) to (24) imply

$$K\phi_{1z}^{\pm} + S_2\omega^{\pm}\phi_0^{\pm} = 0 \quad \text{at} \quad z = 0 \quad , \tag{26}$$

$$\phi_{1z}^{\pm} = 0 \quad \text{at} \quad z = -h \quad . \tag{27}$$

Equations (26) and (27) are of the same form as given in the ray method expansion [4]. Therefore, it follows from (8) that

$$\omega^{\pm} = D(k^{\pm})^2 \quad . \tag{28}$$

The equations for the second approximations yield

$$(K\phi_{2z}^{\pm})_z = -K(k^{\pm})^2\phi_1^{\pm} + \phi_0^{\pm}\nabla \cdot (K\bar{k}) + 2K\bar{k} \cdot \nabla\phi_0^{\pm} + S\psi_1^{\pm}\phi_1^{\pm}$$
$$+ S_1\phi_{0t}^{\pm} - D^{\pm}S_1\phi_0^{(2)}\zeta_t - K[2D^{\pm}\nabla\theta \cdot \nabla\zeta\phi_0^{(2)} - D^{\pm}(\nabla\zeta)^2\phi_0^{(1)}$$

$$- Q(\nabla\zeta)^2 D^+ \phi_0^{(2)}] \quad , \tag{29}$$

$$K\phi_{2z}^{\pm} = -S_2\omega^{\pm}\phi_1^{\pm} - S_2\phi_{0t}^{\pm} + S_2 D^{\pm}\phi_0^{(2)}\zeta_t \quad \text{at} \quad z = 0 \quad , \tag{30}$$

$$\phi_{2z}^{\pm} = \phi_0^{\pm}\nabla h \cdot \bar{k}^{\pm} \quad \text{at} \quad z = -h \quad , \tag{31}$$

where

$$D^{\pm} = \pm(Q' + QP) \quad . \tag{32}$$

The derivation of (29) is a little tedious but straightforward and the details are omitted here. We integrate (29) from $z = -h$ to $z = 0$, and make use of (30), (31) to obtain

$$-S_2\omega^{\pm}\phi_1^{\pm}(x,y,0,t) - S_2\phi_{0t}^{\pm} - \phi_0^{\pm}K(x,y,-h)\bar{k}^{\pm} \cdot \nabla h + S_2 D^{\pm}\zeta_t\phi_0^{(2)} \quad ,$$

$$= \int_{-h}^{0}[(S_1\omega^{\pm} - K(k^{\pm})^2)\phi_1^{\pm} + \phi_0^{\pm}\nabla \cdot (K\bar{k}^{\pm}) + 2K\bar{k}^{\pm} \cdot \nabla\phi_0^{\pm} + S_1\phi_{0t}^{\pm}]dz$$

$$-\int_{-h}^{0}\{D^{\pm}S_1\zeta_t\phi_0^{(2)} + K[D^{\pm}2\nabla\theta \cdot \nabla\zeta\phi_0^{(2)} - D^{\pm}(\nabla\zeta)^2\phi_0^{(1)} - D^{+}Q(\nabla\zeta)^2\phi_0^{(2)}]\}dz. \tag{33}$$

Let

$$\phi_i^{\pm} = R(\zeta)A_i^{\pm}, \; 2RR' = R^2 D^+ Q^{-1} \quad . \tag{34}$$

It is shown in Appendix A that (33) may be reduced to the same transport equation (13)

$$[I(A_0^{\pm})^2]_t + \nabla \cdot \int_{-h}^{0} 2K(A_0^{\pm})^2\bar{k}^{\pm}dz + 2N^{\pm}(A_0^{\pm})^2 = 0 \quad , \tag{35}$$

and by integration along a ray, we have

$$I(A_0^{\pm})^2 J^{\pm}(t) \exp(-\int_{t_0}^{t} 2N^{\pm}I^{-1}dt) = \text{constant} \quad ,$$

or by (34),

$$I(R)^{-2}(\phi_0^{\pm})^2 J^{\pm}(t) \exp(-\int_{t_0}^{t} 2N^{\pm}I^{-1}dt) = \text{constant} \quad . \tag{36}$$

Based upon (32), (34) and (36), we may choose suitable $P$, $Q$ and $R$ so that $\phi_0^{\pm}$ tend to a finite limit at an anomaly.

## 4. Applications

First we consider a line of zero depth $\Gamma : h = 0$, assumed to be smooth. We choose $\mu = 1$, $Q(\zeta) = 1$. Since $A$ behaves like $\zeta^{-1/2}$ at $\Gamma$, as shown in the Appendix B, we choose $R = \zeta^{1/2}$. Then by (32), (34), $P(\zeta)[\log (R^2/Q)]' = \zeta^{-1}$. The comparison equation (16) becomes

$$V''(\beta\zeta) + (\beta\zeta)^{-1}V'(\beta\zeta) - V(\beta\zeta) = 0 \quad . \tag{37}$$

The solutions of (37) may be expressed in terms of modified Bessel functions of zeroth order. We choose $V(\beta\zeta) = I_0(\beta\zeta)$ so that $V(\beta\zeta)$ is regular at $\zeta = 0$ which corresponds to $h \neq 0$. In Appendix B, it will be shown that $\phi_0^{\pm}$ indeed tend to a finite limit as $h \to 0$. From (15) to (19),

$$\phi = \exp(-\beta\theta)[\phi^{(1)}I_0(\beta\zeta) + \phi^{(2)}I_0'(\beta\zeta)] \quad , \tag{38}$$

$$\theta = (S^+ + S^-)/2, \quad \zeta = (S^- - S^+)/2 \quad ,$$

$$\phi^{(1)} = (\phi^+ + \phi^-)/2, \quad \phi^{(2)} = \phi^+ - \phi^-)/2 \quad .$$

If we have two lines of zero depth in a fluid region, corresponding to $\zeta = 0$, $\zeta = \zeta_1$, we may choose $\mu = 0$, $Q(\zeta) = 1$, $R(\zeta) = [\zeta(\zeta_1-\zeta)]^{1/2}$. Then it follows that $P(\zeta) = [\zeta(\zeta_1-\zeta)]'/[\zeta(\zeta_1-\zeta)]$. (16) now becomes

$$[\zeta(\zeta_1-\zeta)V'(\zeta)]' - \beta^2\zeta(\zeta_1-\zeta)V(\zeta) = 0 \quad .$$

At a strictly convex caustic, we follow Ludwig [10] and choose $\mu = 2/3$, $Q = \zeta^{1/2}$, $P = 0$. From (32), (34), it follows that $R = \zeta^{1/4}$ and (16) becomes

$$V''(\beta^{2/3}\zeta) - \beta^{2/3}\zeta V(\beta^{2/3}\zeta) = 0 \quad ,$$

which is the Airy equation. We may choose $V(\beta^{2/3}\zeta) = Ai(\beta^{2/3}\zeta)$ to construct a uniform asymptotic expansion.

In case we have a line of zero depth corresponding to $\zeta = 0$ and a caustic corresponding to $\zeta = \zeta_1$, we take $Q = (\zeta_1-\zeta)^{1/2}$, $P = \zeta^{-1}$, $\mu = 0$ and $R = \zeta^{1/2}(\zeta_1-\zeta)^{1/4}$. Then (16) becomes

$$V''(\zeta) + \zeta^{-1}V'(\zeta) - \beta^2(\zeta_1-\zeta)V(\zeta) = 0 \quad .$$

Finally, if there are two caustics corresponding to $\zeta = 0$ and $\zeta = \zeta_1$, then (16) assumes the form

$$V''(\zeta) - \beta^2\zeta(\zeta_1-\zeta)V = 0 \quad ,$$

and $R = \zeta^{1/4}(\zeta_1-\zeta)^{1/4}$.

## Appendix A. Derivation of the Transport Equation

Here we show that under the conditions (34), (33) is reduced to (13). By comparing (33) with (12), we see that the only difference between them is the extra term on the right-hand side of (33). We multiply both sides of (33) by $2\phi_0^{\pm}$, and make use of (34) and the derivation of (13) from (12) to obtain

$$R^2\{[I(A_0^{\pm})^2]_t + \nabla \cdot \int_{-h}^{0} 2K(A_0^{\pm})^2\, \bar{k}^{\pm} d\zeta + 2N^{\pm}(A_0^{\pm})^2\}$$

$$= 2RR'\{-[I(A_0^{\pm})^2]\zeta_t - \int_{-h}^{0} 2K(A_0^{\pm})^2\, \bar{k}^{\pm} d\zeta \cdot \nabla\zeta$$

$$\pm QA_0^{\pm} \int_{-h}^{0} [S_1\zeta_t A_0^{(2)}\zeta_t + K(2\nabla\theta \cdot \nabla\zeta A_0^{(2)} - (\nabla\zeta)^2 A_0^{(1)} \mp Q(\nabla\zeta)^2 A_0^{(2)})]d\zeta$$

$$\pm 2QA_0^{\pm}A_0^{(2)} S_2\zeta_t\} \quad , \tag{A.1}$$

where

$$A_0^{(1)} = (A_0^{+} + A_0^{-})/(2Q),\ A_0^{(2)} = (A_0^{+} - A_0^{-})/2Q) \quad . \tag{A.2}$$

If we can show that the right-hand side of (A.1) vanishes, then (A.1) is reduced to (13). To this end, we first take the upper sign in the right-hand side of (A.1), and by (19) we replace $\nabla\theta$ by $(\bar{k}^{+} + \bar{k}^{-})/2$, $\nabla\zeta$ by $(\bar{k}^{-} - \bar{k}^{+})/(2Q)$ and $\zeta_t$ by $(\omega^{+} - \omega^{-})/(2Q)$. Then by (A.2) we collect the coefficients of $A_0^{+}$ and $A_0^{-}$, which vanish identically because of (28). Similarly, the right-hand side of (A.1) corresponding to the lower sign also vanishes by the same derivation. Thereore, (35) and (36) follows.

## Appendix B. Regularity at a line of zero depth

To be definite, we assume that $K$, $S_1$ and $S_2$ are smooth, bounded and positive at $\Gamma : h = 0$. By (8) and (9) and the expression for $N(x,y,t)$ in (13), it is easily shown that $N = O(h)$ near $\Gamma$. Since $I > 0$, we see from (34) and (36) that $\phi^{\pm}$ behave like $R[J^{\pm}(t)]^{-1/2}$. Since $R = \zeta^{1/2}$, we show $J^{\pm}(t) = O(\zeta)$ so that $R[J^{\pm}(t)]^{-1/2}$ tend to a finite limit as $\zeta \to 0$. For convenience, a coordinate system $(t,\xi,n)$ is chosen where $\xi$, $n$ are respectively distances along $\Gamma$ and in the normal direction of $\Gamma$ toward the fluid region. Since $\xi$, $n$ are independent of $t$, the Jacobian of transformation from $t$, $\xi$, $n$ - system to $t$, $x$, $y$ - system is given by

$$J\left(\frac{t,x,y}{t,\xi,n}\right) = 1 - \kappa n \quad , \tag{B.1}$$

where $\kappa$ is the curvature of $\Gamma$. In terms of $t$, $\xi$ and $n$, (28) becomes

$$S_\xi^2(1-\kappa n)^{-2} + S_n^2 = -D^{-1}S_t \tag{B.2}$$

where we drop the $\pm$ signs. Based uponn the assumptions on $K$, $S_1$ and $S_2$, it follows from (9) that

$$D = \int_{-h}^{0} K\, d\zeta/I = D_0 h + O(h^2) \quad , \tag{B.3}$$

where $D_0 \neq 0$ at $\Gamma$. We further assume $\partial h/\partial n = \gamma > 0$ at $\Gamma$ and $h = \gamma n + O(n^2)$. Therefore, we can express (A.3) as

$$D = nf(\xi,n) \tag{B.4}$$

where $f(\xi,0) > 0$. From (A.2) and (A.5) we have

$$nS_\xi^2(1-\kappa n)^{-2} + (n^{1/2}S_n)^2 = -f^{-1}S_t \quad . \tag{B.5}$$

Now we introduce a new variable $\eta = n^{1/2}$, and in terms of it, (A.5) becomes

$$\eta^2 S_\xi^2(1-\kappa\eta)^{-2} + S_\eta^2/4 = -f^{-1}S_t \quad , \tag{B.6}$$

and

$$S_\eta = \pm 2[-f^{-1}S_t - \eta^2 S_\xi^2(1-\kappa\eta^2)^{-2}]^{1/2} \quad . \tag{B.7}$$

(B.7) may be solved by the method of characteristics if we assume $\omega = \omega_0 > 0$ and $S$ is sufficiently smooth, on $\eta = 0$, $t > 0$. It is not

difficult to show from the characteristic equations that the Jacobian of transformation from the ray coordinates $t, t_0, \xi_0$ to the $t, \xi, \eta$ - coordinates is given by

$$J\left(\frac{t,\xi,\eta}{t,t_0,\xi_0}\right) = [\omega_0 f(\xi,0)]^{1/2} \quad , \tag{B.8}$$

where $\sigma_1 = t_0$, $\sigma_2 = \xi_0$ is a point on $\eta = 0$, $t > 0$. The Jacobian in (36) may be expressed as

$$J^{\pm}(t) = J\left(\frac{t,x,y}{t,\sigma_1,\sigma_2}\right) = J\left(\frac{t,x,y}{t,\xi,n}\right) J\left(\frac{t,\xi,\eta}{t,t_0,\xi_0}\right) dn/d\eta \quad . \tag{B.9}$$

From (A.1), (A.8) and (A.9), we see that

$$J^{\pm}(t) \sim \mp 2[\omega_0 f(\xi,0)]^{1/2}\eta \quad . \tag{B.10}$$

Now from (A.7),

$$S_\eta(t,\xi,\eta) = \pm 2[\omega_0 f(\xi,0)]^{1/2} + O(\eta^2) \quad .$$

By integration,

$$S^{\pm}(t,\xi,\eta) = S_0 \mp 2[\omega_0 f(\xi,0)]^{1/2}\eta + O(\eta^3) \quad ,$$

and it follows from (19) that

$$\zeta = (S^- - S^+)/2 = 2[\omega_0 f(\xi,0)]^{1/2}\eta + O(\eta^3) \quad . \tag{B.11}$$

By (A.10) and (A.11), we have

$$J^{\pm}(t) \sim \mp\zeta \quad ,$$

and $\phi_0^{\pm}$ tend to a finite limit as $\zeta \to 0$.

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A general uniform asymptotic expansion is constructed to remove certain anomalies which appear in the ray method expansion for unsteady flow in unconfined aquifers, and makes the asymptotic method based upon the principles of geometrical optics even more useful. In particular, expansions uniform in a region containing a caustic, a line of zero depth, a combination of both, two caustics, and two lines of zero depth are explicitly constructed.

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